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SM3 2.2: Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that polynomials of at least first degree have at least one complex root. This means that we can take a polynomial of degree n and rewrite it as a product of a linear factor, which has form $(ax-b)$, and a polynomial of degree $n-1$. That process can be repeated until the polynomial is rewritten as the product of linear factors.

$$f(x) = x^n + x^{n-1} + x^{n-2} + \cdots$$

$$f(x) = (a_1x + b_1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots)$$

$$f(x) = (a_1x + b_1)(a_2x + b_2)(x^{n-2} + x^{n-3} + x^{n-4} + \cdots)$$

$$\vdots$$

$$f(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3)\cdots(a_nx + b_n)$$

<u>Example</u>: Find the complete linear factorization for $x^3 - 3x^2 + 7x - 21$

$(x^3 - 3x^2) + (7x - 21)$	Because 1 to -3 has the same ratio as 7 to -21, this is an ideal grouping problem.
$x^2(x-3) + 7(x-3)$	Factor the GCF from each group.
$(x-3)(x^2+7)$	Rewrite as product of linear factors and the rest of the polynomial.

Are all of the factors linear? Not yet, we'll need to further factor.

$(x-3)\left(x+i\sqrt{7}\right)\left(x-i\sqrt{7}\right)$	Complex factoring

<u>Example</u>: Find the complete linear factorization for $-5x^4 - 15x^2 - 10$

$-5(x^4 + 3x^2 + 2)$	Factor the GCF from the polynomial
$-5(x^2+2)(x^2+1)$	Quadratic techniques
$-5\left(x+i\sqrt{2}\right)\left(x-i\sqrt{2}\right)(x^2+1)$	Complex factoring
$-5\left(x+i\sqrt{2}\right)\left(x-i\sqrt{2}\right)(x+i)(x-i)$	Complex factoring

<u>Example</u>: Find the complete linear factorization for $64x^3 - 27$

$(4x-3)(16x^2+12x+9)$	Sum/Difference of cubes $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
$16(4x-3)\left(x+\frac{3}{8}-\frac{3i\sqrt{3}}{8}\right)\left(x+\frac{3}{8}+\frac{3i\sqrt{3}}{8}\right)$	Quadratic formula $16x^2 + 12x + 9$ has roots $x = -\frac{3}{8} \pm \frac{3i\sqrt{3}}{8}$ *Don't forget to put the leading coefficient out in front of the complex factoring.

Sometimes, you'll know one or more of the roots or factors before you begin, and can use that to your advantage by performing synthetic division using the known root to determine the factorization.

Example: Find the complete linear factorization for each polynomial with the given factor(s).

$x^3 - 39x - 70; (x + 5)$

$$\begin{array}{r|rrrr} \boxed{-5} & 1 & 0 & -39 & -70 \\ & \downarrow & -5 & 25 & 70 \\ \hline & 1 & -5 & -14 & \boxed{0} \end{array}$$

Synthetic division with $x = -5$

$(x + 5)(x^2 - 5x - 14)$

Because $\frac{x^3-39x-70}{x+5} = x^2 - 5x - 14$, we know that $(x + 5)$ and $(x^2 - 5x - 14)$ are the factors of $x^3 - 39x - 70$.

$(x + 5)(x - 7)(x + 2)$

Factor the trinomial

Example: Find the complete linear factorization for each polynomial with the given factor(s).

$x^4 - 6x^3 + 12x^2 - 24x + 32; (x - 2i)$

Setting the known factor $(x - 2i) = 0$ and solving shows that $x = 2i$ is a root of the polynomial. Synthetic division can show us the remaining factor that contains the other roots of the polynomial.

$$\begin{array}{r|ccccc} \boxed{2i} & 1 & -6 & 12 & -24 & 32 \\ & \downarrow & 2i & -4-12i & 24+16i & -32 \\ \hline & 1 & -6+2i & 8-12i & 16i & \boxed{0} \end{array}$$

Synthetic division with $x = 2i$. When the divisor is complex, space terms apart widely.

Because the root $x = 2i$ is not real, its conjugate must also be a root.

$$\begin{array}{r|cccc} \boxed{-2i} & 1 & -6+2i & 8-12i & 16i \\ & \downarrow & -2i & 12i & -16i \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

Synthetic division with $x = -2i$. Use the quotient from the previous division as the new dividend.

Having divided by $(x - 2i)$ and then $(x + 2i)$, the remaining factor is $x^2 - 6x + 8$.

$(x - 2i)(x + 2i)(x^2 - 6x + 8)$ Factor by synthetic division

$(x - 2i)(x + 2i)(x - 4)(x - 2)$ Factor the trinomial

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SM3 2.2: Fundamental Theorem of Algebra

Problems: Find the complete linear factorization for each polynomial.

1) $x^2 + 16$

2) $135x^3 + 40$

3) $x^2 + 2x - 63$

4) $8x^3 - 1$

5) $2x^2 - 7x - 15$

6) $125x^3 + 64$

7) $4x^3 + 12x^2 + 9x$

8) $7x^3 + 21x^2 - 6x - 18$

9) $12x^2 + 26x + 12$

10) $30x^3 - 6x^2 - 30x + 6$

11) $x^3 + 81x$

12) $18x^3 + 42x^2 + 3x + 7$

13) $x^4 + 5x^2 + 6$

14) $x^4 - 64$

15) $x^4 - 13x^2 + 36$

16) $36x^{44} - x^{43} - 21x^{42}$

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Find the complete linear factorization for each polynomial with the given factor(s).

17) $x^3 + 4x^2 + x - 6; (x + 2)$

18) $x^3 - 7x^2 + 2x + 40; (x - 5)$

19) $6x^3 - 29x^2 + 23x + 30; (x - 3)$

20) $15x^3 - 37x^2 - 86x - 24; (3x + 1)$

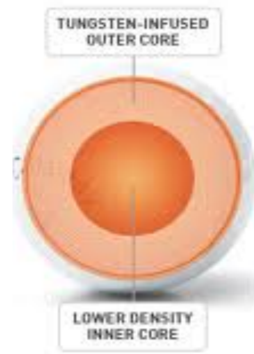
21) $2x^3 + 7x^2 + 32x + 112; (x + 4i)$

22) $3x^3 - 20x^2 + 42x - 20; (3x - 2)$

23) $3x^3 - 6x^2 - 255x - 462; (x - 11)$

24) $10x^3 + 25x^2 + 40x + 100; (x - 2i)$

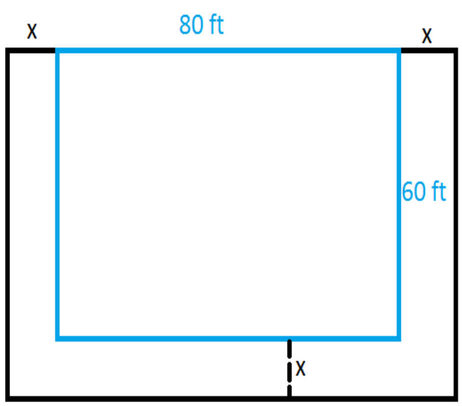
25) Austin's golf ball design company, *Hit Our Balls and Yell "Fore!"*, has finished a new style of ball. The inner core of the ball is a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ that varies in radius to accommodate players with different types of swings. This core is suspended within the outer casing and then the ball is filled with tungsten-infused material that makes up the outer core. The ball, itself, has a radius of 1 inch.



a) Determine the volume of the entire ball.

b) Build polynomial $v_{oc}(r)$ to describe the volume of the outer core in terms of the radius of the inner core.

c) Factor $v_{oc}(r)$ completely.



26) After graduating from high school, Jamison goes on to a lucrative career in engineering, as inspired by his math teacher. To say "thanks", he decides to build his math teacher a rectangular wave pool on 7488 ft^2 of space that was previously used for something unimportant, like student parking. This pool will have a length of 60ft and a width of 80ft. The size of the walkway around the pool is uniform but unknown. The walkway does not completely surround the pool; you do not want swimmers hopping in where the waves are generated!

a. Construct polynomial $A(x)$ that describes the surface area being made into the pool with the walkway.

b. Find the value of x.